1 Original Mathematical Program

We seek an optimal solution to the following (non-linear integer) mathematical program:

$$
\begin{aligned}
&(\text{Total weighted affinity}) && \max_x \sum_{c\in C}\sum_{i\in I}\sum_{j\in I} a_{ij}x_{ic}x_{jc}\omega_i\omega_j && \\
&(\text{Each guest assigned to exactly one table}) && \text{s.t.} \sum_{c\in C} x_{ic} = 1 && \forall i \in I \\
&(\text{Table capacity}) && \sum_{i\in I} x_{ic} \le \tau_c && \forall c \in C \\
&(\text{Must-link constraints}) && x_{ic} = x_{i'c} && \forall c \in C, (i,i') \in P_O \\
&(\text{Cannot-link constraints}) && x_{ic} + x_{i'c} \le 1 && \forall c \in C, (i,i') \in P_N \\
&(\text{Binary assignment of cluster}) && x_{ic} \in \{0,1\} && \forall i \in I; c \in C
\end{aligned}
$$

where:

- $I = \{1, \cdots, n\}$: set of n guests
- $C = \{1, \cdots, K\}$: set of K tables
- τ_c : capacity of table c
- a_{ij} : affinity between guest i and j
- $P_O, P_N \subset I \times I$: pairs of guests that must and cannot be together
- ω_i : weight of guest i

and:

- x_{ij} : binary variable indicating if guest i is assigned to table j

2 Linearized Mathematical Program

We use McCormick inequalities to linearize the non-linear bilinear terms $x_{ic}x_{jc}$ in the objective

$$
\begin{array}{lll}
\text{(Total weighted affinity)} & \max\limits_{x,w} \sum\limits_{c \in C} \sum\limits_{i \in I} \sum\limits_{j \in I} a_{ij} w_{ijc} \omega_i \omega_j & \\
\text{(Each guest assigned to exactly one table)} & \text{s.t.} \sum\limits_{c \in C} x_{ic} = 1 & \forall i \in I \\
\text{(Table capacity)} & \sum\limits_{i \in I} x_{ic} \leq \tau_c & \forall c \in C \\
\text{(Must-link constraints)} & x_{ic} = x_{jc} & \forall c \in C; (i,j) \in P_O \\
\text{(Cannot-link constraints)} & x_{ic} + x_{jc} \leq 1 & \forall c \in C; (i,j) \in P_N \\
\text{(McCormick 1)} & w_{ijc} \leq x_{ic} & \forall i,j \in I; c \in C \\
\text{(McCormick 2)} & w_{ijc} \leq x_{jc} & \forall i,j \in I; c \in C \\
\text{(McCormick 3)} & w_{ijc} \geq x_{ic} + x_{jc} - 1 & \forall i,j \in I; c \in C \\
\text{(Binary assignment of cluster)} & x_{ic} \in \{0,1\} & \forall i \in I; c \in C \\
\text{(Non-negative linearizing variables)} & w_{ijc} \geq 0 & \forall i,j \in I; c \in C
\end{array}
$$

3 Linearized Mathematical Program with grouped guests

We can substitute the must-link constraints directly in the objective and replace I by $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}''$ where $\mathcal{I}' = \{\{i\} : i \in I; (i,j) \notin P_0, \forall j \in I\}$ represents the guests that are not part of any must-link constraints and $\mathcal{I}'' = \{\{i_1, \cdots, i_{|k|}\} : i_l \in I \cap K, l = 1, \cdots, |K|, K \in \mathcal{F}\}$ represents "meta guests" or clusters of guests that must be seated together. $\mathcal{F}$ is the collection of trees in the forest formed by the induced subgraph with edges P_0 and nodes having at least one endpoint in P_0. Note that $\mathcal{I}'$ and $\mathcal{I}'$ are now treated as collections of sets and $\mathcal{I}', \mathcal{I}'' \subset P(I)$.

If we denote $\bar{n} = |\mathcal{I}|$, then the problem has less decision variables than the previous since $\bar{n} < n$ if $|P_O| > 1$. In other words if we must merge at least 2 guests, then we have at least one less "meta guest" or cluster of guests to assign to tables.

Once we have grouped guests into a new meta-guest we must modulate the affinity between that cluster and the other cluster guests. In our simple case, this is simply the average affinity:

$$\bar{a}_{KL} = \frac{\sum_{i \in K} \sum_{j \in L} a_{ij}}{\bar{\omega}_K \bar{\omega}_L} K \in \mathcal{I}, L \in \mathcal{I} \tag{1}$$

$$\tag{2}$$

where for meta guest $K \in \mathcal{I}$, $\bar{\omega}_K = \sum_{i \in K} \omega_i$ represents the total weights of the guests in that group. The table capacities must be modified by considering: $n_K = |K|$, the number of guests in cluster K, where $\sum_{K \in \mathcal{I}} n_K = n$.

The cannot link constraints are replaced by a new set of constraints P'_N where 2 clusters cannot be grouped together as soon as there exists one guest in each that cannot be seated together.

$$
\begin{array}{llll}
(\text{Total weighted affinity}) & \max_{x,w} & \sum_{c\in C}\sum_{I\in\mathcal{I}'}\sum_{J\in\mathcal{I}'} \bar{a}_{IJ} w_{IJc}\bar{\omega}_I\bar{\omega}_J & \\
(\text{Each guest assigned to exactly one table}) & \text{s.t.} & \sum_{c\in C} x_{Ic} = 1 & \forall I \in \mathcal{I}' \\
(\text{Table capacity}) & & \sum_{I\in\mathcal{I}'} n_I x_{Ic} \le \tau_c & \forall c \in C \\
(\text{Cannot-link constraints}) & & x_{Ic} + x_{Jc} \le 1 & \forall c \in C; (I,J) \in P'_N \\
(\text{McCormick 1}) & & w_{IJc} \le x_{Ic} & \forall I,J \in \mathcal{I}; c \in C \\
(\text{McCormick 2}) & & w_{IJc} \le x_{Jc} & \forall I,J \in \mathcal{I}; c \in C \\
(\text{McCormick 3}) & & w_{IJc} \ge x_{Ic} + x_{Jc} - 1 & \forall I,J \in \mathcal{I}; c \in C \\
(\text{Binary assignment of cluster}) & & x_{Ic} \in \{0,1\} & \forall I \in \mathcal{I}; c \in C \\
(\text{Non-negative linearizing variables}) & & w_{IJc} \ge 0 & \forall I,J \in \mathcal{I}; c \in C
\end{array}
$$

if we wish to compute the true optimal value, we may add the constant $\sum_{(i,j)\in P_O^k} a_{ij}\omega_i\omega_j$ (the total affinity of guests that must be placed together), but this is immaterial to finding the optimal solution since this is constant regardless of the assignment.